Axisymmetric Flow of an Ideal Incompressible Fluid about a Solid Torus

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E. Sternberg and M. A. Sadowsky

A Technical Report to the

Office of Naval Research
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Summary

This paper contains an exact solution in series form to the problem presented by the irrotational, axisymmetric flow of an ideal, incompressible fluid past a solid torus of circular cross section. At infinity the fluid is assumed to be in a state of uniform motion parallel to the axis of the torus. The solution is based on the use of toroidal coordinates, and is given in terms of Legendre functions of fractional order as well as complete and incomplete elliptic integrals of the first and second kind. The individual component solutions employed here are interpreted physically through their relation to the basic ring singularities represented by the source ring and the vortex ring.

The problem is first approached via Stokes' stream function exclusively and is subsequently re-solved independently in terms of the velocity potential alone. The convergence of the pertinent series developments is found to be unusually favorable, and a complete stream line pattern, corresponding to an illustrative numerical example, is included.

Introduction

Potential problems involving toroidal boundaries have received repeated attention in the literature. Apparently the first investigation of this kind is due to Carl Neumann [1][1] who dealt with the distribution of temperature in a solid torus and with the analogous problem in electrostatics. The corresponding axisymmetric problem in ideal fluid flow was first investigated by Hicks [2] and was reconsidered later by Dyson [3], who also examined certain unsymmetric flow cases associated with the torus. The rigorous treatment of allied problems in the theory of elasticity is more recent; Freiberger [4] disposed of the problem of pure torsion of a torus of circular cross section, whereas the case of pure bending was treated by the current authors [5]. It is this last investigation which motivated the present study, and the analytical prerequisites needed here were established to a considerable extent in connection with the work described in [5].

Although the hydrodynamic investigations cited previously yielded formal solutions characterizing the irrotational axisymmetric flow of an inviscid, incompressible fluid past a torus of circular centerline and circular cross section, the problem appears to be in need of further attention. Hicks [2] based his analysis on toroidal coordinates and toroidal harmonics but did not deal with the determination of the stream function, which is of primary physical interest. Dyson [3], on the other hand, arrived at a representation of the potential and stream functions in terms of series of definite integrals; the significance of this representation for numerical evaluations of the results appears to be rather limited. In view of

[1] Numbers in brackets refer to the bibliography at the end of the paper.

the currently available numerical tabulations of the Legendre functions of fractional order [6], a re-examination of the problem would seem to serve a useful purpose.

In 1948 Weinstein [7] discussed axially symmetric potential flows and, in particular, considered an extension of the method of sources and sinks to cases in which the singularities are no longer confined to the axis of symmetry. This paper stimulated further interest in related problems. Thus, Streeter [8], with the aid of ring and disk singularities, obtained streamline patterns corresponding to flows around various blunt-nosed half-bodies and annular-shaped bodies; his approach is based on a semi-numerical procedure. At the same time elliptic integral representations were established [9] for the potential and stream functions appropriate to rotationally symmetric distributions of sources and vorticity over circular rings and disks. Concurrently, Van Tuyl [10] deduced a similar analytical representation of the flow generated by a uniform source disk, and applied it to the study of a new family of half-bodies. Still more recently, Streeter [11] arrived at the flow around a torus of nearly circular cross section by superposition of a ring doublet upon a uniform stream. This solution will later be identified as a first approximation to the rigorous solution for the torus of exactly circular cross section; moreover, the results to be developed here are free from the indirect trial-and-error features inherent in Streeter's approach. Reference should also be made to a paper by Schiffman and Spencer [12] in which toroidal coordinates are used to study the flow about a lens-shaped object.

In what follows we examine certain pertinent sequences of singular solutions of the governing equations referred to toroidal coordinates. The physical significance of these component solutions, in terms of which the solution to the torus problem will ultimately be expanded, is investigated with particular attention to the presence or absence of a circulation in the corresponding potential and stream functions: it is shown how the foregoing component solutions may be generated through successive limit processes applied to the basic axisymmetric ring singularities of the source ring and the vortex ring. This part of the paper may be of interest beyond the present application. The torus problem is then formulated and solved on the basis of the stream-function approach. A second, independent solution, resting on the potential-function approach, is indicated merely since it provides further insight into the problem under consideration. In order to render the present exposition sensibly self-contained, some of the material given in [5] has also been briefly included here.

The Governing Equations. Toroidal Coordinates

For the sake of convenience we summarize at this place the basic equations governing the steady irrotational flow of an ideal incompressible fluid in the presence of axial symmetry. With reference to the cylindrical coordinates (ρ, γ, z), where the z-axis is assumed coincident with the axis of symmetry, the velocity potential $\phi(\rho, z)$ satisfies Laplace's equation in the form,

$$\frac{\partial}{\partial \rho}(\rho \phi_\rho) + \frac{\partial}{\partial z}(\rho \phi_z) = 0 \qquad (1)^2$$

and the velocity components are given by

$$v_\rho = \phi_\rho, \qquad v_z = \phi_z \qquad (2)$$

Stokes' stream function $\psi(\rho, z)$ is introduced through the relations,

[2] Subscripts attached to functions which originally bear no subscript denote differentiation with respect to the argument indicated.

$$\psi_z = -\rho\phi_\rho, \qquad \psi_\rho = \rho\phi_z \tag{3}$$

and, alternatively, admits the representation,

$$\psi(\rho,z) = \int_{(\rho_0,z_0)}^{(\rho,z)} \left[\rho\phi_z \, d\rho - \rho\phi_\rho \, dz\right] \tag{4}$$

in which (ρ_0, z_0) are the coordinates of an arbitrary fixed point in the meridional half-plane $\rho \geq 0$. In view of Equation (1), which may be regarded as an integrability condition for Equations (3), the line integral (4) is independent of the path, and $\psi(\rho,z)$ is single-valued, in any simply connected domain throughout which $\phi(\rho,z)$ is regular. As was emphasized by Weinstein [7], the single-valuedness of ψ can no longer be inferred if the domain of regularity of ϕ is multiply connected. Similarly, and under analogous circumstances, a single-valued stream function may give rise to a multiple-valued velocity potential. A potential function and a stream function which are related according to Equations (3) will henceforth be called conjugate with respect to each other.

It follows from Equations (1), (3) that ψ conforms to the "stream equation",

$$\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\psi_\rho\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\psi_z\right) = 0 \tag{5}$$

and Equations (2), (3) further imply,

$$v_\rho = -\frac{1}{\rho}\psi_z, \qquad v_z = \frac{1}{\rho}\psi_\rho \tag{6}$$

A specific velocity field determines its associated potential as well as the conjugate stream function only within an arbitrary additive constant. Finally, we recall that the curves $\psi(\rho,z) = \text{constant}$ are the stream lines and that $2\pi(\psi_2 - \psi_1)$ constitutes the total flow between the stream surfaces generated by $\psi = \psi_1$ and $\psi = \psi_2$.

In what follows we shall have occasion to refer to toroidal coordinates (α, β, γ) which are defined through the transformation,

$$x = \frac{\bar{q}}{q - p} \cos \gamma, \qquad y = \frac{\bar{q}}{q - p} \sin \gamma, \qquad z = \frac{\bar{p}}{q - p} \tag{7}$$

where (x,y,z) are the Cartesian coordinates and

$$p = \cos \alpha, \qquad \bar{p} = \sin \alpha, \qquad q = \cosh \beta, \qquad \bar{q} = \sinh \beta \tag{8}$$

The ranges of the curvilinear coordinates are given by

$$0 \leq \alpha \leq 2\pi, \qquad 0 \leq \beta < \infty, \qquad 0 \leq \gamma \leq 2\pi \tag{9}$$

so that

$$-1 \leq p, \bar{p} \leq 1, \qquad 1 \leq q < \infty, \qquad 0 \leq \bar{q} < \infty \tag{10}$$

Furthermore, with the auxiliary notation,

$$\mu = (q - p)^{1/2} \tag{11}$$

we have,

$$\rho = (x^2 + y^2)^{1/2} = \frac{\bar{q}}{\mu^2} \tag{12}$$

Equations (7), (8), (11) imply

$$(\rho - \frac{q}{\bar{q}})^2 + z^2 = \frac{1}{\bar{q}^2}, \qquad \rho^2 + (z - \frac{p}{\bar{p}})^2 = \frac{1}{\bar{p}^2} \tag{13}$$

The coordinate surfaces α = constant are thus spherical bowls of radius $|\bar{p}|^{-1}$ centered on the z-axis at $z = p/\bar{p}$. The surfaces β = constant are tori, the meridional cross sections of which are the circles of radius $\bar{q}^{-1}$ centered on the ρ-axis at $\rho = q/\bar{q}$. As $\beta \to \infty$ the tori degenerate into the circle[3] $\rho = 1$, $z = 0$. Figure 1 shows the traces of

[3]This normalization of the coordinate transformation (7) is, of course, unessential.

the foregoing two mutually orthogonal families of coordinate surfaces upon the meridional half-planes γ = constant.

The differential of arc-length is defined by

$$ds^2 = \left(\frac{d\alpha}{h_1}\right)^2 + \left(\frac{d\beta}{h_2}\right)^2 + \left(\frac{d\gamma}{h_3}\right)^2 \tag{14}$$

and the metric coefficients here appear as

$$h_1 = h_2 = \mu^2, \qquad h_3 = \frac{\mu^2}{\bar{q}} \tag{15}$$

A routine computation now yields for Laplace's Equation (1) in toroidal coordinates,

$$\mu^2\bar{q}(\phi_{\alpha\alpha} + \phi_{\beta\beta}) - \bar{p}\bar{q}\phi_\alpha + (1 - pq)\phi_\beta = 0 \tag{16}$$

whereas the stream Equation (5) becomes

$$\mu^2\bar{q}(\psi_{\alpha\alpha} + \psi_{\beta\beta}) + \bar{p}\bar{q}\,\psi_\alpha - (1 - pq)\,\psi_\beta = 0 \tag{17}$$

The toroidal transform of Equations (3) is

$$\phi_\alpha = -\frac{\mu^2}{\bar{q}}\,\psi_\beta, \qquad \phi_\beta = \frac{\mu^2}{\bar{q}}\,\psi_\alpha \tag{18}$$

and the toroidal components of the velocity vector assume the form,

$$\left.\begin{aligned} v_\alpha &= \mu^2\phi_\alpha = -\frac{\mu^4}{\bar{q}}\,\psi_\beta \\ v_\beta &= \mu^2\phi_\beta = \frac{\mu^4}{\bar{q}}\,\psi_\alpha \end{aligned}\right\} \tag{19}$$

Toroidal Potential and Stream Functions[4]

In this section we consider certain aggregates of solutions to Laplace's Equation (16) and to the stream equation (17), in toroidal coordi-

[4]See [13], Chapter X, for a discussion of toroidal harmonics. Toroidal stream functions are also dealt with in [5].

nates. Neither of these two equations is separable in the strict sense. There do exist, however, pseudo-product solutions of the form,

$$\phi(\alpha,\beta) = \mu A(\alpha)B(\beta) \tag{20}$$

to Laplace's equation, whereas the stream equation admits the pseudo-product solutions,

$$\psi(\alpha,\beta) = \frac{\bar{q}^2}{\mu} C(\alpha)D(\beta) \tag{21}$$

Substitution of Equations (20), (21) into Equations (16), (17), subsequent separation of variables, and adherence to the restriction that the solutions sought have a period 2π in α, ultimately yield,

$$\left.\begin{aligned} \phi(\alpha\beta) &= \mu\left[\cos n\alpha \text{ or } \sin n\alpha\right]\left[P_{n-1/2}(q) \text{ or } Q_{n-1/2}(q)\right] \\ \psi(\alpha\beta) &= \frac{\bar{q}^2}{\mu}\left[\cos n\alpha \text{ or } \sin n\alpha\right]\left[P'_{n-1/2}(q) \text{ or } Q'_{n-1/2}(q)\right] \\ &(n = 0, 1, 2, \ldots) \end{aligned}\right\} \tag{22}$$[5]

where $P_{n-1/2}$ and $Q_{n-1/2}$ are the Legendre functions of the first and second kind, respectively.

For future reference, we cite Legendre's equation,

$$\bar{q}^2 P''_{n-1/2} + 2qP'_{n-1/2} - (n^2 - \tfrac{1}{4})P_{n-1/2} = 0 \tag{23}$$

and recall the recursion formulas,

$$\left.\begin{aligned} P_{-n-1/2} &= P_{n-1/2} \\ 4nqP_{n-1/2} &= (2n+1)P_{n+1/2} + (2n-1)P_{n-3/2} \\ 2\bar{q}^2 P'_{n-1/2} &= (2n-1)(qP_{n-1/2} - P_{n-3/2}) \end{aligned}\right\} \tag{24}$$

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[5] The primes denote differentiation with respect to the argument q.

Equations (24) are valid for all integral values of n, and remain valid if $P_{n-1/2}$ is replaced with $Q_{n-1/2}$. We also note the bilinear identity,

$$P'_{n-1/2}Q'_{n-3/2} - P'_{n-3/2}Q'_{n-1/2} = -\frac{2n-1}{2\bar{q}^2} \tag{25}$$ [6]

Furthermore, we shall need to refer to Laplace's integral,

$$P_{-1/2}(q) = \frac{1}{\pi}\int_0^{\pi} \frac{d\alpha}{\sqrt{q + \bar{q}\cos\alpha}} \tag{26}$$ [7]

and to the integral representation,

$$Q_{n-1/2}(q) = \frac{1}{\sqrt{2}}\int_0^{\pi} \frac{\cos n\alpha}{\mu}\, d\alpha \tag{27}$$ [8]

which gives rise to the Fourier expansion,

$$\frac{1}{\mu} = \frac{\sqrt{2}}{\pi} Q_{-1/2} + \frac{2\sqrt{2}}{\pi}\sum_{n=1}^{\infty} Q_{n-1/2}\cos n\alpha \tag{28}$$ [9]

where μ is defined in Equation (11).

Equations (26), (27) supply the link to the representation of the Legendre functions of fractional index in terms of elliptic integrals. Thus, let

$$k = \sqrt{\frac{2}{q+1}} = \frac{1}{\cosh \beta/2}, \qquad k' = \sqrt{1-k^2} = \frac{\bar{q}}{q+1} \tag{29}$$

Setting $\theta = \alpha/2$ in Equation (26), we arrive at

$$P_{-1/2} = \frac{2k}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{2k}{\pi} K' \tag{30}$$

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[6] See [13], art. 45.

[7] See [13], p. 381.

[8] See [14], p. 443.

[9] The argument of $P_{n-1/2}$ and $Q_{n-1/2}$ is henceforth assumed to be q, unless otherwise specified.

and setting $\theta = (\pi - \alpha)/2$ in Equation (27), we reach,

$$Q_{-1/2} = kK \tag{31}$$

where K and K' are the complete elliptic integrals of the first kind referred to the modulus k and the complementary modulus k', respectively. Alternatively, we may define moduli,

$$k_1 = \sqrt{\frac{2\bar{q}}{q + \bar{q}}}, \qquad k_1' = \sqrt{1 - k_1^2} = \frac{1}{q + \bar{q}} = e^{-\beta} = \frac{r_2}{r_1} \tag{32}$$

in which

$$\left.\begin{aligned} r_1 &= \left[(\rho + 1)^2 + s^2\right]^{1/2} = \frac{\sqrt{2}\, e^{\beta/2}}{\mu} \\ r_2 &= \left[(\rho - 1)^2 + s^2\right]^{1/2} = \frac{\sqrt{2}\, e^{-\beta/2}}{\mu} \end{aligned}\right. \tag{33}$$

The new moduli are related to k and k', introduced in Equation (29), through the Landen transformation,[10]

$$k_1 = \frac{2\sqrt{k'}}{1 + k'}, \qquad k_1' = \frac{1 - k'}{1 + k'} \tag{34}$$

so that

$$\left.\begin{aligned} K_1 &= (1 + k')K' = \frac{k}{\sqrt{k_1'}} K' \\ E_1 &= (k' - 1)K' + \frac{2}{1 + k'} E' \end{aligned}\right\} \tag{35}$$

(35)[11]

Here K_1 and E_1 are the complete elliptic integrals of the first and second kind based on k_1. The moduli k_1, k_1' are identical with those

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[10] See [14], p. 507.

[11] The second of Equations (35) is obtained by differentiating the first with respect to k'.

previously employed by Hicks [2], Fouquet [15], and the current authors [9], [5].

Symmetric and Antisymmetric Flow Fields

Among the potential and stream functions listed in Equations (22), those involving the Legendre functions of the second kind become infinite as $q \to 1$, and therefore violate the regularity requirements on the z-axis inherent in the torus problem to be considered subsequently. We shall thus limit the present discussion to the potential and stream functions involving $P_{n-1/2}$ and its derivative, which remain analytic for every finite q and vanish at Cartesian infinity; they become singular on the "limit circle", as $q \to \infty$.

We may distinguish between two basic flow cases according as the velocity field is symmetric or antisymmetric with respect to the plane $z = 0$, that is, according as the velocity potential is an even or odd function of z and α. Symmetric flows are generated by the potentials and stream functions,

$$\left.\begin{aligned} \Phi_n &= \mu P_{n-1/2} \cos n\alpha \\ \Psi_n &= \frac{\bar{q}^2}{\mu} P'_{n-1/2} \sin n\alpha \\ &(n = 0, 1, 2, \ldots) \end{aligned}\right\} \qquad (36)$$

It is important to observe that Φ_n and Ψ_n do not constitute a conjugate pair in the sense of Equations (18). We now seek to establish the stream function $\overline{\Psi}_n$ and the velocity potential $\overline{\Phi}_n$ which are conjugate to Φ_n and Ψ_n, respectively. To this end we confirm with the aid of Equations (18) that,

$$\left.\begin{aligned} \overline{\Psi}_{n+1} - \overline{\Psi}_n &= \frac{2}{2n+1}(\Psi_{n+1} - \Psi_n) \\ \overline{\Phi}_{n+1} - \overline{\Phi}_n &= \frac{2n+1}{2}(\Phi_{n+1} - \Phi_n) \\ &(n = 0, 1, 2, \ldots) \end{aligned}\right\} \tag{37}$$

Since $\Psi_o = 0$, we may take $\overline{\Phi}_o$ to be zero, and the foregoing recursion relations yield

$$\left.\begin{aligned} \overline{\Psi}_n &= \overline{\Psi}_o + 2\sum_{i=1}^{n} \frac{1}{2i-1}(\Psi_i - \Psi_{i-1}) \\ \overline{\Phi}_n &= \frac{1}{2}\sum_{i=1}^{n} (2i-1)(\Phi_i - \Phi_{i-1}) \\ &(n = 1, 2, 3, \ldots) \end{aligned}\right\} \tag{38}$$

In order to complete the determination of $\overline{\Psi}_n$ for $n = 0, 1, 2, \ldots$, we need to find $\overline{\Psi}_o$. For this purpose we introduce the parameter t through

$$\operatorname{am}(t,k) = \theta = \frac{\pi - \alpha}{2} \tag{39}$$

or, equivalently, through

$$t = F(\theta,k) = \int_0^{\theta} \frac{d\vartheta}{(1 - k^2 \sin^2 \vartheta)^{1/2}} \tag{40}$$

$\operatorname{am}(t,k)$ being the amplitude function and $F(\theta,k)$ the incomplete elliptic integral of the first kind, both referred to the modulus k given in Equations (29). Thus,

$$\operatorname{sn}t = \sin\theta = \cos\frac{\alpha}{2}, \qquad \operatorname{cn}t = \cos\theta = \sin\frac{\alpha}{2}, \qquad \operatorname{dn}t = \frac{k\mu}{\sqrt{2}} \tag{41}$$

where sn, cn, dn denote the Jacobian elliptic functions[12] for the modulus k.

According to Equations (30), (36), and (41), we may write

$$\Phi_o = \mu P_{-1/2} = \frac{2\sqrt{2}}{\pi} K' \, \mathrm{dnt} \tag{42}$$

Equations (18), applied to the conjugate pair $\{\Phi_o, \overline{\Psi}_o\}$, by virtue of Equation (42), now lead to

$$\left.\begin{aligned} \frac{\partial\overline{\Psi}_o}{\partial\alpha} &= \frac{\sqrt{2}}{\pi}\left[\frac{E'}{\mathrm{dnt}} - k^2K' \frac{\mathrm{cn}^2 t}{\mathrm{dn}^3 t}\right] \\ \frac{\partial\overline{\Psi}_o}{\partial\beta} &= -\frac{\sqrt{2}\, k^2 k'}{\pi} K' \frac{\mathrm{snt}\ \mathrm{cnt}}{\mathrm{dn}^3 t} \end{aligned}\right\} \tag{43}$$

where E' is the complete elliptic integral of the second kind based on the modulus k'. Equations (43) are explicitly integrable by recourse to the identities listed on p. 516 of [14]. There results

$$\overline{\Psi}_o = \frac{2\sqrt{2}}{\pi}\left[(K' - E')t + k^2K' \frac{\mathrm{snt}\ \mathrm{cnt}}{\mathrm{dnt}} - K'E(t)\right] \tag{44}$$ [13]

in which E(t) designates the incomplete elliptic integral of the second kind for the modulus k. We note from Equations (38) that $\overline{\Psi}_n$ and $\overline{\Phi}_n$, which are conjugate to the pseudo-product solutions Φ_n and Ψ_n of the potential and stream equations, no longer possess the pseudo-product structure. Indeed, a single potential function Φ_i gives rise to a conjugate stream function $\overline{\Psi}_i$ whose representation requires an infinite series of stream functions in pseudo-product form; the closed representation given by

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[12] See [14], Chapter XXII, for a discussion of Jacobian elliptic functions and integrals.

[13] It should be noted that the Jacobian elliptic functions appearing here may be eliminated by means of Equations (41).

Equations (38), (44) was made possible only through the introduction of elliptic functions and integrals.

Since

$$\left.\begin{aligned} \mathrm{am}(t + 2K) &= \mathrm{am}\,t + \pi \\ E(t + 2K) &= E(t) + 2E \end{aligned}\right\} \quad (45)$$

and recalling Legendre's identity,

$$EK' + E'K - KK' = \frac{\pi}{2} \quad (46)$$

as well as the periodicity properties of the Jacobian elliptic functions, we conclude from Equations (44), (38) that

$$\left.\begin{aligned} \overline{\Psi}_n(\alpha + 2\pi, \beta) - \overline{\Psi}_n(\alpha, \beta) = 2\sqrt{2} \\ (n = 0, 1, 2, \ldots) \end{aligned}\right\} \quad (47)$$

The single-valued potentials Φ_n thus correspond to associated conjugate stream functions $\overline{\Psi}_n$ which are many-valued for $-\infty < \alpha < \infty$, or else discontinuous along the cut $\alpha = 0$ in the meridional half-plane corresponding to $0 \leq \alpha \leq 2\pi$, $0 \leq \beta < \infty$. This result is consistent with our previous observations regarding the singular character of Φ_n as $q \to \infty$. On the other hand, the conjugate sequences $\left\{\overline{\Phi}_n, \Psi_n\right\}$ $(n = 1, 2, 3, \ldots)$ are both single-valued.

The treatment of the anti-symmetric flow fields generated by

$$\left.\begin{aligned} \phi_n &= \mu P_{n-1/2} \sin n\alpha \\ \psi_n &= \frac{\bar{q}^2}{\mu} P'_{n-1/2} \cos n\alpha \end{aligned}\right\} \quad (48)$$

is strictly analogous to the preceding discussion of symmetric fields, and we may confine ourselves to stating the pertinent results. If $\{\phi_n, \overline{\Psi}_n\}$ and $\{\overline{\phi}_n, \Psi_n\}$ are conjugate pairs of potential and stream functions, then

$$\left.\begin{aligned} \overline{\Psi}_{n+1} - \overline{\Psi}_n &= \frac{-2}{2n+1}(\Psi_{n+1} - \Psi_n) \\ \overline{\phi}_{n+1} - \overline{\phi}_n &= -\frac{2n+1}{2}(\phi_{n+1} - \phi_n) \\ &(n = 0, 1, 2, \ldots) \end{aligned}\right\} \qquad (49)$$

and consequently,

$$\left.\begin{aligned} \overline{\Psi}_n &= -2\sum_{i=1}^{n} \frac{1}{2i-1}(\Psi_i - \Psi_{i-1}) \\ \overline{\phi}_n &= \overline{\phi}_o - \tfrac{1}{2}\sum_{i=1}^{n}(2i-1)(\phi_i - \phi_{i-1}) \\ &(n = 1, 2, 3, \ldots) \end{aligned}\right\} \qquad (50)$$

Here we may take $\overline{\Psi}_o = 0$ since ϕ_o vanishes, whereas $\overline{\phi}_o$ turns out to be given by

$$\overline{\phi}_o = \frac{\sqrt{2}}{\pi}\left[(K' - E')t - K'E(t)\right] \qquad (51)$$

The functions ϕ_n, Ψ_n, and $\overline{\Psi}_n$ are periodic with period 2π in α. On the other hand, we find that

$$\left.\begin{aligned} \overline{\phi}_n(\alpha + 2\pi, \beta) - \overline{\phi}_n(\alpha, \beta) &= \sqrt{2} \\ &(n = 0, 1, 2, \ldots) \end{aligned}\right\} \qquad (52)$$

which indicates a circulation in the velocity potentials $\overline{\phi}_n$ conjugate to the single-valued stream functions Ψ_n.

We now turn to a physical interpretation of the potential and stream functions discussed in this section. With regard to symmetric flows, we note on the basis of Equation (3.9) of Reference [9] that the potential of a uniform source ring of total strength m and unit radius (coincident with the limit circle of the toroidal coordinate system) in our present notation appears as

$$\phi = \frac{2m}{\pi} \frac{K_1}{r_1} \tag{53}$$

with r_1 and K_1 defined as in Equations (33), (35). A trivial computation, involving Equations (53), (35), (33), (32), and (30), identifies

$$\phi = \frac{m}{\sqrt{2}} \Phi_o, \qquad \psi = \frac{m}{\sqrt{2}} \overline{\Psi}_o \tag{54}$$

as the potential and stream functions appropriate to the source ring under consideration. Moreover, we confirm[14] with the aid of Equation (47) that the stream function possesses a cyclic period of 2m, as is consistent with hydrodynamic theory.

In a similar manner, guided by Equation (7.5) of Reference [9], we recognise

$$\phi = \frac{\Gamma}{\sqrt{2}} \bar{\phi}_o, \qquad \psi = \frac{\Gamma}{\sqrt{2}} \psi_o \tag{55}$$

as the potential and stream functions belonging to the antisymmetric flow generated by a uniform vortex ring of unit radius and total circulation Γ. The circulation constant Γ of the velocity potential is readily verified by means of Equation (52).

Equations (54), (55) establish the significance of the initial members in the sequences of symmetric and antisymmetric flows discussed previously.

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[14] See [9], Equation (5.7).

The remaining members in these two aggregates of singular flows, which are characterized by singularities of progressively higher order on the limit circle, may be generated through successive z-differentiations applied to the basic singular solutions which represent the source ring and the vortex ring. Indeed, a somewhat tedious computation, depending on the repeated use of the Legendre relations (24), yields the result,

$$\left.\begin{aligned} \frac{\partial \Phi_n}{\partial z} &= -\frac{(2n-1)}{4}\phi_{n-1} + n\phi_n - \frac{2n+1}{4}\phi_{n+1} \\ \frac{\partial \psi_n}{\partial z} &= -\frac{2n+1}{4}\Psi_{n-1} + n\Psi_n - \frac{(2n-1)}{4}\Psi_{n+1} \\ &(n = 0, 1, 2, \ldots) \end{aligned}\right\} \quad (56)[15]$$

Moreover, these recursion formulas remain valid if Φ and ϕ, as well as Ψ and ψ, are interchanged while the signs of the right-hand members are reversed. As an illustrative example we determine the solution corresponding to a ring-doublet, whose potential is the first derivative with respect to z of the potential for a source ring. By Equations (54), (56), (50), the ring-doublet is thus represented by

$$\left.\begin{aligned} \phi &= -\frac{m}{2\sqrt{2}}\phi_1 \\ \psi &= -\frac{m}{2\sqrt{2}}\overline{\psi}_1 = \frac{m}{\sqrt{2}}(\psi_1 - \psi_o) \end{aligned}\right\} \quad (57)$$

and these formulas are readily transformed into the expressions derived by Streeter [11].

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[15] Observe that $\phi_{-1} = -\phi_1$ and $\Psi_{-1} = -\Psi_1$, according to Equations (24), (36), (48).

Solution of the Torus Problem: The Stream-Function Approach

We are now in a position to consider the main problem of this investigation. Let the boundary of the torus coincide with the parametric surface $\beta = \beta_o$ (see Figure 1). Then, according to Equations (13),

$$\Delta = \frac{b}{a} = q_o = \cosh \beta_o \qquad (58)^{16}$$

where a and b are the radii of the cross section and of the centerline, respectively, and Δ will be called the "shape-ratio" of the torus.

The problem consists in determining a stream function $\psi(\alpha,\beta)$ which satisfies Equation (17) throughout the domain $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq \beta_o$, and conforms to the boundary and regularity conditions,

$$\psi(\alpha,\beta_o) = c, \qquad \psi(\alpha,\beta) \rightarrow \psi_\infty = \frac{\rho^2}{2} \text{ as } (\alpha,\beta) \rightarrow (0,0) \qquad (59)$$

where c is the initially unknown value of ψ on the surface of the torus, and ψ_∞ designates the stream function associated with the uniform velocity field $v_\rho = 0$, $v_z = 1$ at infinity.

Guided by the antisymmetric nature of the uniform velocity field at infinity, we assume the desired solution in the form,

$$\psi(\alpha,\beta) = \frac{\rho^2}{2} + \sum_{n=0}^{\infty} a_n \psi_n \qquad (60)$$

with ψ_n defined by the second of Equations (48), and note that $\psi(\alpha,\beta)$ automatically conforms to the condition at infinity since $\psi_n(0,0) = 0$. The coefficients of superposition a_n $(n = 0, 1, 2, \ldots)$ are to be determined consistent with the boundary conditions for $\beta = \beta_o$. By virtue of Equations (60), (59), (48), and (12), this leads to

---

[16] The subscript, zero, attached to any function of β, will henceforth indicate its evaluation for $\beta = \beta_o$.

$$\sum_{n=0}^{\infty} a_n P'_{n-1/2}(q_o) \cos n\alpha = \frac{c\mu_o}{\bar{q}_o^2} - \frac{1}{2\mu_o^3} \tag{61}$$

In order to expand the right-hand member of Equation (61) in a Fourier series, we recall the expansion (28). With the aid of this expansion, and making use of the identities (24) applied to the Legendre functions of the second kind, we thus arrive at

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} a_n P'_{n-1/2}(q_o) \cos n\alpha = \\ &\qquad \frac{2\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \left[\frac{2c}{4n^2-1} + 1\right] \frac{Q'_{n-1/2}(q_o)}{1+\delta_{no}} \cos n\alpha \end{aligned}\right\} \tag{62}$$

in which δ_{no} designates the Kronecker delta. Equation (62) implies

$$a_n = \left(\frac{2c}{4n^2-1} + 1\right) A_n \tag{63}$$

provided

$$A_n = \frac{2\sqrt{2}\, Q'_{n-1/2}(q_o)}{\pi(1+\delta_{no})P'_{n-1/2}(q_o)} \tag{64}$$

and, in accordance with Equation (60), the appropriate velocity potential appears as

$$\phi(\alpha,\beta) = a + \sum_{n=0}^{\infty} a_n \bar{\phi}_n \tag{65}$$

the functions $\bar{\phi}_n$ being defined in Equations (50) and (51).

It is apparent from the preceding developments that the boundary conditions (59) do not suffice to determine the yet unknown value c of ψ on the surface of the torus. This observation is in agreement with the well known fact that the boundary conditions fail to characterize the solution uniquely if the flow domain is multiply connected.[17] It follows from

[17] This statement, of course, in no way contradicts the uniqueness theorems of hydrodynamics, which require also the specification of the initial conditions; see, for example, [16], p. 422.

Equation (52) that the velocity potential represented by Equation (65) is multiple-valued unless c is chosen appropriately. In view of Equation (52), a circulation-free potential is assured if and only if

$$\sum_{n=0}^{\infty} a_n = 0 \tag{66}$$

or, by Equation (63),

$$c = -\frac{1}{2}\left[\sum_{n=0}^{\infty} A_n\right]\left[\sum_{n=0}^{\infty} \frac{An}{4n^2-1}\right]^{-1} \tag{67}$$

The complete solution to our problem is now given by Equations (60), (65), where ψ_n and $\bar{\phi}_n$ are defined in Equations (48), (50), (51), and the coefficients a_n follow from Equations (63), (64), (67). The toroidal velocity components are readily obtained from Equations (60), (48), (12), with the aid of Equations (18):

$$\left.\begin{aligned} v_\alpha &= \frac{pq-1}{\mu^2} + \frac{\mu}{4}\sum_{n=0}^{\infty} a_n\left[2\bar{q}^2 P'_{n-1/2} - (4n^2-1)\mu^2 P_{n-1/2}\right]\cos n\alpha \\ v_\beta &= \frac{-p\bar{q}}{\mu^2} - \frac{\bar{q}\mu}{2}\sum_{n=0}^{\infty} a_n P'_{n-1/2}\left[2n\mu^2 \sin n\alpha + \bar{p}\cos n\alpha\right] \end{aligned}\right\} \tag{68}$$

Finally, we note for future reference, that the series (60) admits the rearrangement,

$$\psi = \frac{\rho^2}{2} + \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n} a_m\right](\psi_n - \psi_{n+1}) \tag{69}$$

<u>Solution of the Torus Problem: The Potential-Function Approach</u>

In this section we sketch a second, independent approach to the problem, in terms of the velocity potential. Although this method of attack is more cumbersome than the one adopted previously, it should nevertheless prove to be instructive. The alternative formulation of the problem as a problem of

Neumann, requires the determination of a function $\phi(\alpha,\beta)$ which is harmonic in the domain $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq \beta_o$, and obeys the boundary and regularity conditions,

$$\left.\frac{\partial\phi}{\partial\beta}\right|_{(\alpha,\beta_o)} = 0, \qquad \phi(\alpha,\beta) \rightarrow \phi_{\infty} = z \quad \text{as} \quad (\alpha,\beta) \rightarrow (0,0) \tag{70}$$

where ϕ_{∞} is the potential of the uniform velocity field at infinity. We thus put

$$\phi(\alpha,\beta) = z + \sum_{n=1}^{\infty} b_n \phi_n \tag{71}$$

with reference to the definition (48) of the potential functions ϕ_n. Applying the first of conditions (70), we reach with the aid of Equation (28),

$$\left.\begin{aligned} &(b_{n+1} - b_n)P'_{n+1/2}(q_o) - (b_n - b_{n-1})P'_{n-3/2}(q_o) = \\ &\qquad\qquad \frac{4\sqrt{2}}{\pi}\left[Q'_{n-3/2}(q_o) - Q'_{n+1/2}(q_o)\right] \\ &\qquad b_o = 0, \quad (n = 0, 1, 2, \ldots) \end{aligned}\right\} \tag{72}$$

Equation (72) is a finite difference equation of the second order for the unknown coefficients of superposition b_n $(n = 1, 2, 3, \ldots)$ which possesses an integrating factor $P'_{n-1/2}(q_o)$.

Now let

$$c_n = (b_n - b_{n-1})P'_{n-1/2}P'_{n-3/2} \tag{73}$$ [18]

and Equation (72) may be written as

[18]The argument of the Legendre functions appearing in this and the following equations is understood to be q_o.

$$c_{n+1} - c_n = \frac{4\sqrt{2}}{\pi}\left[Q'_{n-3/2} - Q'_{n+1/2}\right] P'_{n-1/2} \tag{74}$$

This difference equation of the first order is integrable; by use of the bilinear identity (25), we obtain

$$\left.\begin{aligned} \frac{c_n}{P'_{n-1/2}P'_{n-3/2}} &= b_n - b_{n-1} \\ &= \frac{\sqrt{2}}{\pi}\left\{\left[2n-3+\frac{4\bar{q}^2\lambda}{2n-1}\right]\frac{Q'_{n-3/2}}{P'_{n-3/2}} - \left[2n+1+\frac{4\bar{q}^2\lambda}{2n-1}\right]\frac{Q'_{n-1/2}}{P'_{n-1/2}}\right\} \end{aligned}\right\} \tag{75}$$

in which λ is an arbitrary constant of integration. A second integration, based on Equation (75), yields

$$\left.\begin{aligned} b_n &= -\sum_{m=0}^{n}\left[1+\frac{4\bar{q}^2\lambda}{4m^2-1}\right]A_m - \frac{1}{2}\left[2n-1+\frac{4\bar{q}^2\lambda}{2n+1}\right]A_n \\ &(n = 1, 2, 3, \ldots) \end{aligned}\right\} \tag{76}$$

the notation being that of Equation (64). In order that the series (71) be convergent, it is necessary that

$$\lim_{n\to\infty} b_n = 0 \tag{77}$$

This formal convergence condition, which here takes the place of the single-valuedness condition (66),[19] serves to determine the parameter λ. Indeed, since

$$\lim_{n\to\infty} nA_n = 0 \tag{78}$$

we conclude from Equations (77), (76), that

$$\lambda = \frac{c}{2\bar{q}^2} \tag{79}$$

with c defined by Equation (67), and thus,

---

[19] Recall that the functions ϕ_n entering Equation (71), and the conjugate stream functions Ψ_n are all single-valued.

$$\left.\begin{aligned} b_n &= -\sum_{m=0}^{n} a_m - \frac{2n-1}{2}\, a_n \\ &(n = 1, 2, 3, \ldots) \end{aligned}\right\} \quad (80)$$

the coefficients a_n retaining their previous definition given by Equations (63), (64), and (67). This completes the independent determination of the velocity potential introduced in Equation (71).

With a view toward demonstrating the equivalence of the solutions derived in this and the preceding sections, we note from Equation (80) that

$$\left.\begin{aligned} \sum_{m=0}^{n} b_m &= \frac{a_o}{2} - \frac{2n+1}{2} \sum_{m=0}^{n} a_m \\ (n = 0, 1, 2, \ldots)&; \qquad b_o = 0 \end{aligned}\right\} \quad (82)$$

Consequently, Equation (69) may be written as

$$\psi = \frac{\rho^2}{2} + \sum_{n=0}^{\infty} \left[\frac{a_o}{2n+1} - \frac{2}{2n+1} \sum_{m=0}^{n} b_m\right] (\psi_n - \psi_{n+1}) \quad (83)$$

It now follows from Equations (83), (49) that the conjugate velocity potential appears as

$$\phi = z + \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} b_m - \frac{a_o}{2}\right] (\phi_n - \phi_{n+1}) \quad (84)$$

which, after a permissible rearrangement, assumes the form,

$$\phi = z - \frac{a_o}{2}\,\phi_o + \sum_{n=0}^{\infty} b_n\, \phi_n \quad (85)$$

Since $\phi_o = 0$, Equation (85) is identical with Equation (71). It is interesting to observe that the representation (71) for $\phi(\alpha,\beta)$, in contrast to that given by Equation (65), does not involve any elliptic functions or integrals.

Discussion, Numerical Example

A comparison of Equations (69) or (84) with the second of Equations (57), or of Equation (71) with the first of Equations (57), reveals that the leading term in the series contributions to $\psi(\alpha,\beta)$ and $\phi(\alpha,\beta)$ are precisely the stream and potential functions belonging to the ring-doublet. This observation identifies Streeter's solution [11], for the torus of nearly circular cross section, as a first approximation to the exact solution presented here. The fact that already a one-term approximation corresponds to a torus whose cross section is very close to a circle, is accounted for by the extremely rapid convergence of the series under consideration.

Although the convergence of the representation (69) for the stream function is even more favorable than that given in Equation (60), it was still found to be more expedient to use the latter representation in the numerical example which follows. The reason for this lies in the loss of numerical accuracy caused by the difference formations inherent in Equation (69).

The stream pattern shown in Figure 2 applies to a shape-ratio $\Delta = b/a = 3$. It was obtained by choosing various fixed values of z, and by subsequently selecting α and β judiciously and consistent with the last of Equations (7). For these values of α and β the corresponding values of ψ were computed from Equation (60) with the aid of the numerical tables [6]. A maximum of five terms in the series for ψ was needed in order to reach an accuracy of four significant figures. Once a sufficient number of z-profiles of the stream surface had been established, the stream lines $\psi =$ constant were determined graphically. In order to locate the stagnation point — the point of intersection of the separation stream line with the meridional section of the torus — Equation (68) was solved for $v_\alpha(\alpha,\beta) = 0$. A suitable method of successive approximations revealed that $v_\alpha(\alpha,\beta_o)$ vanishes to four significant figures for $\cos\alpha = 0.121$.

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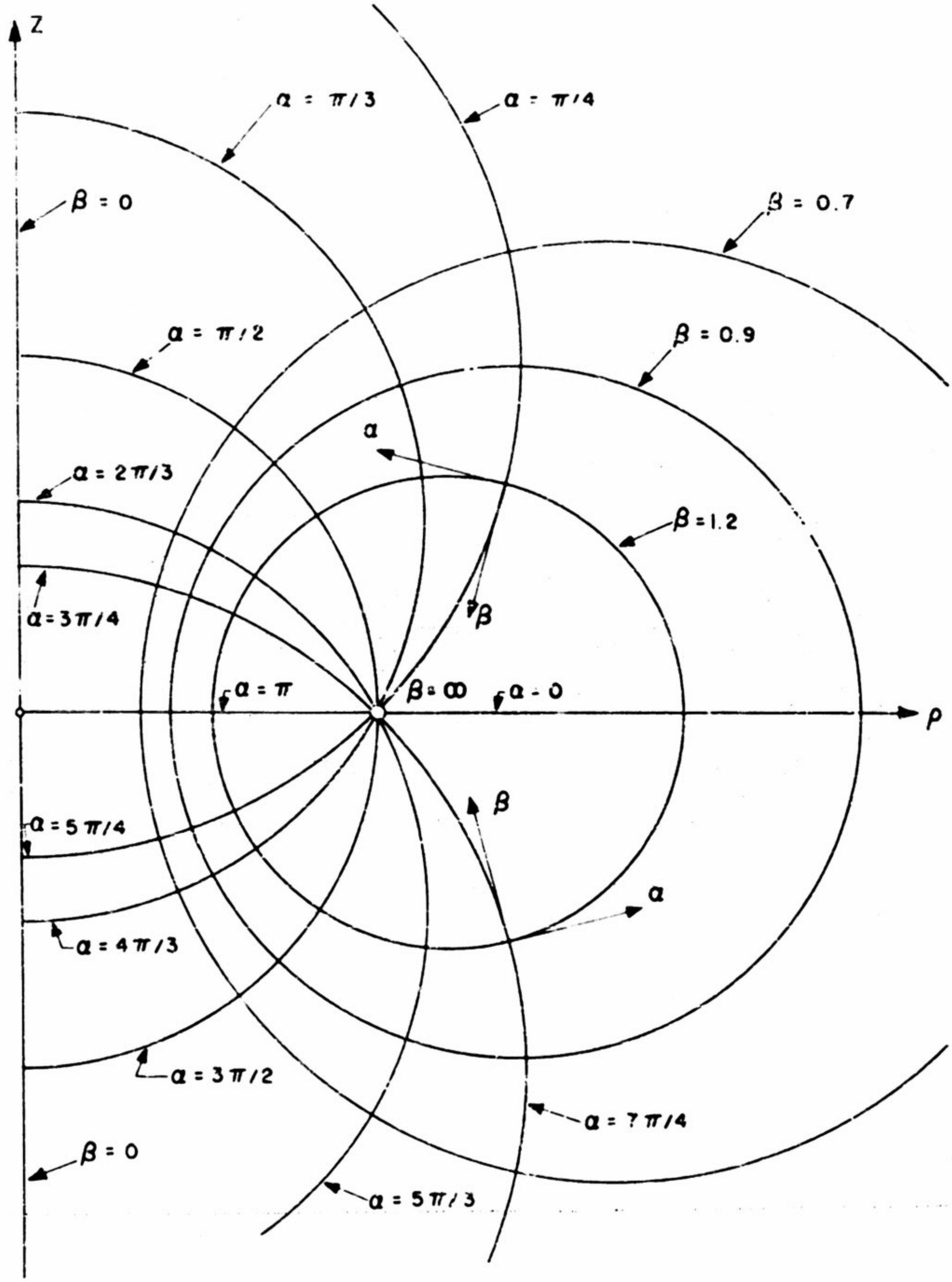

FIG. 1 - TOROIDAL COORDINATES. TRACES OF SURFACES α = CONST. AND β = CONST. ON A HALF-PLANE γ = CONST.

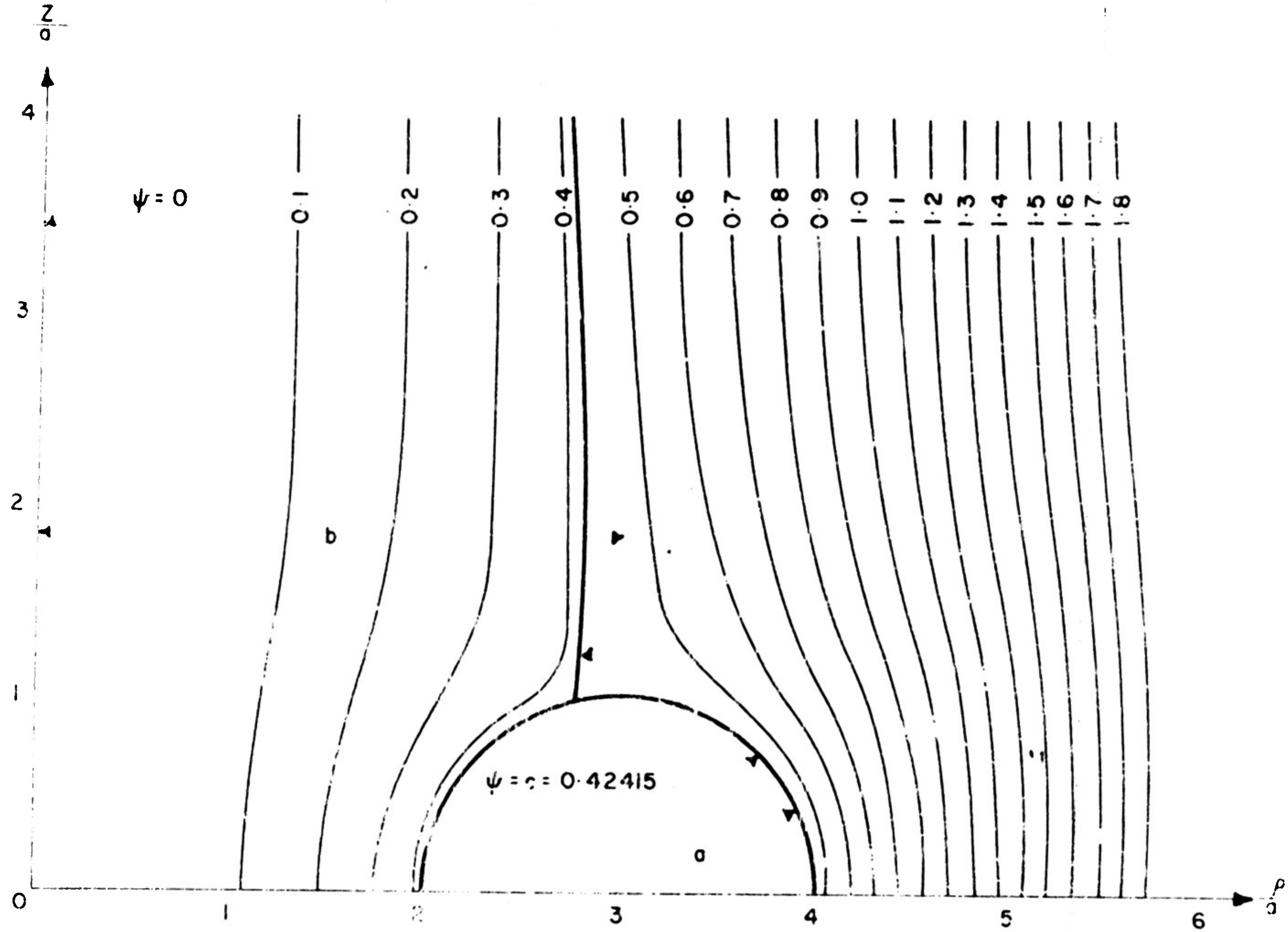

FIG. 2. STREAMLINES FOR $\Delta = b/a = 3$, $V_z = 1$ AT INFINITY

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